Derandomizing Random
Deterministic Walks

Kent Quanrud

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Recall: (s,t)-connectivity in $O(\log n)$ space

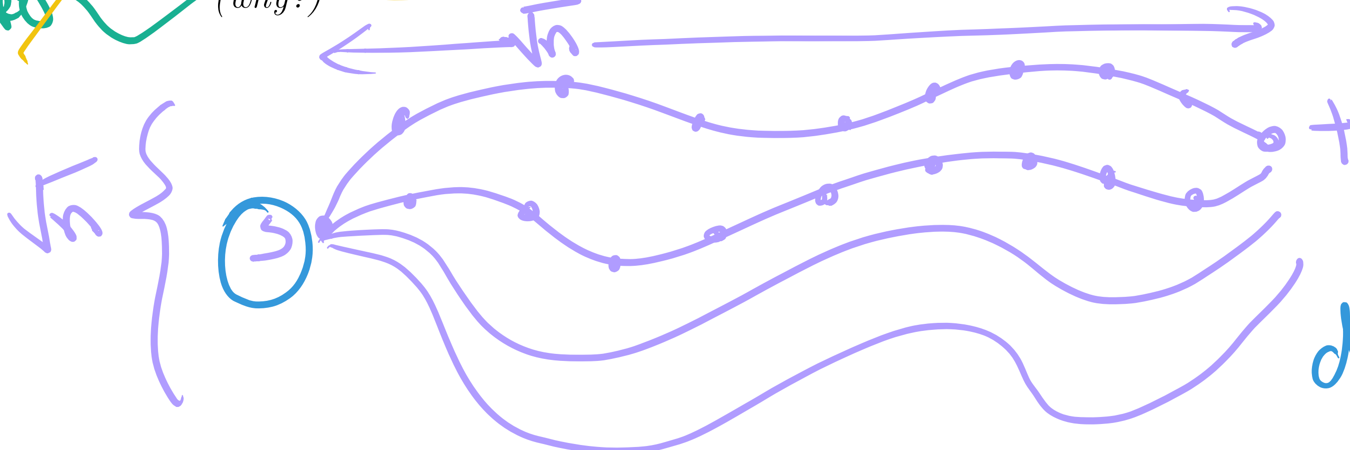
- $G=(V,E)$ undirected, $s,t \in V$: *are s and t connected?* $O(\log n)$ bits
- only allowed $O(\log n)$ space need $2^{O(\log n)} = \mathrm{poly}(n)$
- previously: random walk for $O(mn)$ steps from s

random walk $\xrightarrow{\text{fast}}$ stationary

From the homework:

max degree 6

- If G has constant spectral gap, then diameter is $O(\log n)$. *(why?)*
- If G has constant spectral gap, then deterministic (s,t)-connectivity. *(why?)*



$d \in \mathbb{R}^V$

$$\text{stationary} = \frac{d}{2m} = \bar{d}$$

$x_k \in \Delta^V$ = dist. after k steps

$$\|x_k - \bar{d}\| \le \frac{1}{2^k} n$$

$$k = 5\log(n) \Rightarrow \|x_k - \bar{d}\| \le \frac{1}{n^4}$$

$$|x_k(v) - \underbrace{\bar{d}(v)}_{\ge 1/n^2}| \le \|x_k - \bar{d}\| \le \frac{1}{n^4}$$

$$x_k(v) \ge \frac{1}{n^2} - \frac{1}{n^4} > 0$$

Recall: (s,t)-connectivity in $O(\log n)$ space

- $G = (V, E)$ undirected, $s, t \in V$: *are s and t connected?*
- only allowed $O(\log n)$ space
- previously: random walk for $O(mn)$ steps from s

From the homework:

- If G has constant spectral gap, then diameter is $O(\log n)$. *(why?)*
- If G has constant spectral gap, then determinstic (s,t)-connectivity. *(why?)*

Theorem 1. *There is a $O(\log n)$ space, polynomial time deterministic algorithm for (s,t)-connectivity in undirected graphs with n vertices.*

- Omer Reingold. "Undirected connectivity in log-space". In: *J. ACM* 55.4 (2008), 17:1–17:24. Preliminary version in STOC, 2005.

expander = undirected
constant degree
constant spectral gap

1 High level overview

- Connectivity easy on expanders
- Key idea: *implicitly convert graph to an expander*

- Simplifying assumption: input graph G is regular with constant degree d

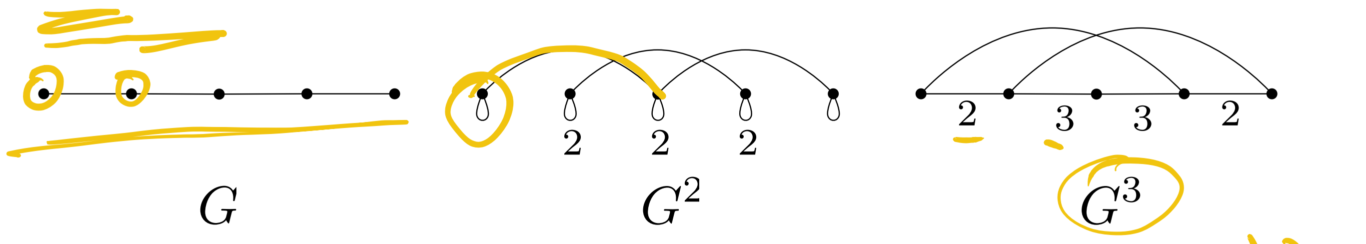
  spectral gap $\geq \frac{1}{n^2}$ (doable with some prepreocessing)

- build up expander with two graph operations:
  1. **powering**
  2. **zig-zag product**

Powering

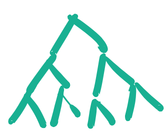
$G^4 = (G^2)^2$

Goal: improve spectral gap



G^k = multi-graph generated by all k-step walks in G $= (V, E = P^k)$

- (u, v) has multiplicity equal to # k-step walks from u to v



$\gamma = \frac{1}{n^2} \leftarrow$

$1 - (1-\gamma)^2 = \frac{2}{n^2} - \frac{1}{n^4}$

$\approx \frac{2}{n^2}$

Lemma.

1. *G^k has n vertices*
2. *G^k is regular with degree d^k.* $\leftarrow$
3. *If random walk on G has spectral gap γ, then random walk on G^k has spectral gap $1 - (1 - \gamma)^k$.*

small constant degree H

1.0.1 Zig-Zag Product

d_0^2 regular

Goal: decrease degree

- $G = (V, E)$ regular graph w/ n vertices and degree d
- $H = (V_0, E_0)$ regular graph w/ d vertices and degree d_0 $= O(1)$
- identify $V_0 = [d]$
- vertices in $H \leftrightarrow$ steps in G

walks in G

Zig-Zag product $\mathcal{Z}(G \mid H)$

Vertices vertex set $V \times [d]$

$k_1, k_2 \in [d_0]$

Edges: (k_1, k_2)th neighbor of $(v_1, i_1) \in V \times [d]$:

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow[1]{} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow[3]{} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow[4]{} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

$v_1 \xrightarrow{i_2} v_2$, $v_2 \xrightarrow{i_3} v_1$

$i_1 \xrightarrow{k_1} i_2 \longrightarrow i_3 \xrightarrow{k_2} i_4$

1. step in H from i_1 to its k_1th neighbor, i_2.
2. moves v_1 to its i_2th neighborm, v_2.
3. move from i_2 to i_3 if v_1 is the i_3th neighbor of v_2.
4. moves i_3 to its k_2th neighbor in H.

(google images)

Notes:

- steps 2, 3 have 0 degrees of freedom
- v_1 adjacent to v_2 in G
- i_1 not (necessarily) adjaceny to i_2 in H ?!
- s, t connected in G only if ... $\begin{pmatrix} s \\ i \end{pmatrix}, \begin{pmatrix} v \\ j \end{pmatrix}$ connected
- $(s, i), (t, j)$ connected in G only if ...

Lemma 2. *Let*

- $G = (V, E)$ *regular undirected graph w/ n vertices and degree d* — γ_G
- H *regular undirected graph w/ d vertices and degree* d_0 — $O(1)$ — γ_H

Then $\mathcal{Z}(G \mid H)$ *has:*

- *nd vertices,* — little bigger
- *degree* d_0^2
- *spectral gap* $\gamma_G \gamma_H^2$*.* — down

$\gamma \in [0,1]$

$\gamma_H = \frac{3}{4}$

$\gamma_G \to \gamma_G \gamma_H^2$ $\quad \gamma_H = \frac{3}{4}$ $\quad \gamma_H^2 = \frac{9}{16}$

Lemma 1. Power

1. G^k *has* n *vertices*

2. G^k *is regular with degree* d^k.

3. *If random walk on* G *has spectral gap* γ, *then random walk on* G^k *has spectral gap* $1-(1-\gamma)^k$.

Lemma 2. *Let*

- $G = (V, E)$ *regular undirected graph w/* n *vertices and degree* d
- H *regular undirected graph w/* d *vertices and degree* d_0

Then $\mathcal{Z}(G \mid H)$ *has:*

- nd *vertices,*
- *degree* d_0^2 ←
- *spectral gap* $\gamma_G \gamma_H^2$.

$$G \xrightarrow[d\uparrow]{\gamma\uparrow} G^2 \xrightarrow[d\downarrow]{\gamma\downarrow} \mathcal{Z}(G^2|H)$$

$$\xrightarrow[d\leftrightarrow]{\gamma\uparrow}$$

1.1 Completing the proof

Lemma 3. *Let* G *be a regular undirected graph with* n *vertices, degree* d^2, *and spectral gap* γ_G. *Let* H *be a regular undirected graph with* d^4 *vertices, degree* d, *and spectral gap* γ H. *Then* $\mathcal{Z}\left(G^2 \mid H\right)$ *is a regular undirected graph with* $d^4 n$ *vertices, degree* d^2, *and spectral gap* $\left(1-(1-\gamma_G)^2\right)\gamma_H^2$.

Lemma 4. *Let G be a regular undirected graph with n vertices, degree d^2, and spectral gap γ_G. Let H be a regular undirected graph with d^4 vertices, degree d, and spectral gap γH. Then $\mathcal{Z}(G^2 \mid H)$ is a regular undirected graph with $d^4 n$ vertices, degree d^2, and spectral gap $\left(1-(1-\gamma_G)^2\right)\gamma_H^2$.*

$\leftarrow \geq 3/4$

Deterministic connectivity

Input

$s, t \in V$

1. G regular with n vertices, constant degree d^2, spectral gap $\geq 1/\operatorname{poly}(n)$
2. H with d^4 vertices, degree d, $\geq$ ~~3/4~~

$\gamma_H \geq \frac{3}{4}$

(explicit construction)

γ

	# vertices	degree
$G_0 = G$	n	d^2
$G_1 = \mathcal{Z}(G_0^2, \mathcal{H})$	$d^4 n$	d^2
$G_2 = \mathcal{Z}(G_1^2, \mathcal{H})$	$d^8 n$	d^2
$G_k = \mathcal{Z}(G_{k-1}^2, \mathcal{H})$ $\quad (1.089)^k/n^2$	$d^{4k} n$	

$\geq \frac{1}{16}$ for $k = \log(n)$ (poly(n) for $k = \log(n)$)

$\gamma_G \leq \frac{1}{16}$

$(1-(1-\gamma_G)^2)\gamma_H^2$

$\geq (2\gamma_G - \frac{1}{16}\gamma_G)\left(\frac{3}{4}\right)^2 \geq 1.089\,\gamma_G \quad 1.01\gamma_G$

$= \frac{31}{16}\gamma_G \cdot \frac{9}{16}$

vertices

Space analysis

Claims.

1. *The space required to simulate a step on G_j^2 is $O(1)$ plus the space required to simpulate a step on G_j.*
2. *The space required to simulate a step on $\mathcal{Z}\big(G_j^2 \mid H\big)$ is $O(1)$ plus the space required to simulate a step on G_j^2.*

If above hold, then space required to simulate step on G_k is $O(k + \log(n))$, as desired.

$G_k = \mathcal{Z}(G_{k-1}^2, H)$

G_j^2

input:
vertex v_1
(i, j)

① query G_j for ith neighbor of v_1, $\Rightarrow v_2$

② query G_j for jth neighbor of v_2, $\Rightarrow v_3$

$\mathcal{Z}(G_k^2, H)$

input: (v_1, i_1), (k_1, k_2)

$i_1 \xrightarrow{k_1} i_2$ // query H, $O(1)$

$v_1 \xrightarrow{i_2} v_2$ // $O(1)$ + query(G_k^2)

i_3

// try all i_3
$O(1)$ + query(G_k^2)

$i_3 \xrightarrow{k_2} i_4$

// $O(1)$

2 Analysis of the zig-zag product

Lemma 2. *Let*

- $G = (V, E)$ *regular undirected graph w/* n *vertices and degree* d
- H *regular undirected graph w/* d *vertices and degree* d_0

Then $\mathcal{Z}(G \mid H)$ *has:*

- nd *vertices,*
- *degree* d_0^2
- *spectral gap* $\gamma_G \gamma_H^2$.

2.1

References

[Rei08] Omer Reingold. "Undirected connectivity in log-space". In: *J. ACM* 55.4 (2008), 17:1–17:24. Preliminary version in STOC, 2005.

[Vad12] Salil P. Vadhan. "Pseudorandomness". In: *Found. Trends Theor. Comput. Sci.* 7.1-3 (2012), pp. 1–336.

small constant degree H

1.0.1 Zig-Zag Product

d_0^2 regular

Goal: decrease degree

- $G = (V, E)$ regular graph w/ n vertices and degree d
- $H = (V_0, E_0)$ regular graph w/ d vertices and degree d_0 $= O(1)$
- identify $V_0 = [d]$
- vertices in $H \leftrightarrow$ steps in G

walks in G

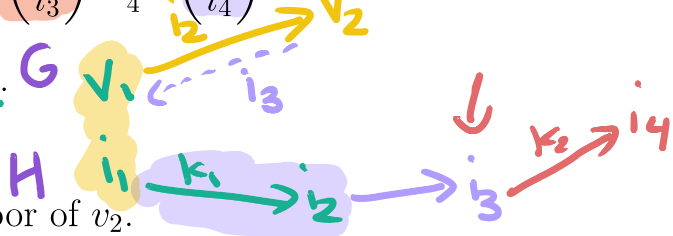
Zig-Zag product $\mathcal{Z}(G \,|\, H)$

Vertices vertex set $V \times [d]$ $V \times V_H$

$k_1, k_2 \in [d_0]$

Edges: (k_1, k_2)th neighbor of $(v_1, i_1) \in V \times [d]$:

$$\binom{v_1}{i_1} \xrightarrow{1} \binom{v_1}{i_2} \xrightarrow{2} \binom{v_2}{i_2} \xrightarrow{3} \binom{v_2}{i_3} \xrightarrow{4} \binom{v_2}{i_4}$$



$I \otimes R_H$ 1. step in H from i_1 to its k_1th neighbor, i_2.

Z 2. moves v_1 to its i_2th neighborm, v_2.

3. move from i_2 to i_3 if v_1 is the i_3th neighbor of v_2.

$I \otimes R_H$ 4. moves i_3 to its k_2th neighbor in H.

(google images)

Notes:

- steps 2, 3 have 0 degrees of freedom
- v_1 adjacent to v_2 in G
- i_1 not (necessarily) adjaceny to i_2 in H ?!
- s, t connected in G only if ... $\binom{s}{i}, \binom{v}{j}$ connected
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Lemma 2. *Let*

- $G = (V, E)$ *regular undirected graph w/ n vertices and degree d*
- H *regular undirected graph w/ d vertices and degree* d_0

Then $\mathcal{Z}(G \mid H)$ *has:*

- *nd vertices,*
- *degree* d_0^2
- *spectral gap* $\gamma_G \gamma_H^2$.

R_G = random walk on G

R_H = random walk on H

I_G = "identity" walk on G

Random step in $Z(G|H)$

$$(I_G \otimes R_H) Z (I_G \otimes R_H)$$

$S: \mathbb{R}^d \to \mathbb{R}^d \quad (d = V_H)$

$Sv = \frac{\mathbb{1}}{d} \qquad v \in \Delta^d$

(random walk on a clique + 1 self-loop)

$R_H \approx S$

$$\underline{(I_G \otimes S) Z (I_G \otimes S)}$$

$$\begin{pmatrix} v_1 \\ \vdots \\ i_1 \end{pmatrix} \xrightarrow[(I\otimes S)]{} \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ i_2 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ \vdots \\ i_2 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ \vdots \\ i_3 \end{pmatrix}}_{Z} \xrightarrow[(I\otimes S)]{} \begin{pmatrix} v_2 \\ \vdots \\ i_4 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ \vdots \\ i_1 \end{pmatrix} \xrightarrow[R_G \otimes S]{} \begin{pmatrix} v_2 \\ \vdots \\ i_4 \end{pmatrix} = $$

One random step in R_G

+ one random in S

$$(R_G \otimes S)$$

$$(I_G \otimes R_H) Z (I_G \otimes R_H)$$

$$\underset{\gamma_H}{\approx} (I_G \otimes S) Z (I_G \otimes S)$$

$$= (R_G \otimes S) \approx R_G$$

$\otimes$ Tensor product of graphs

$G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$

d_1-regular $\quad$ d_2-regular

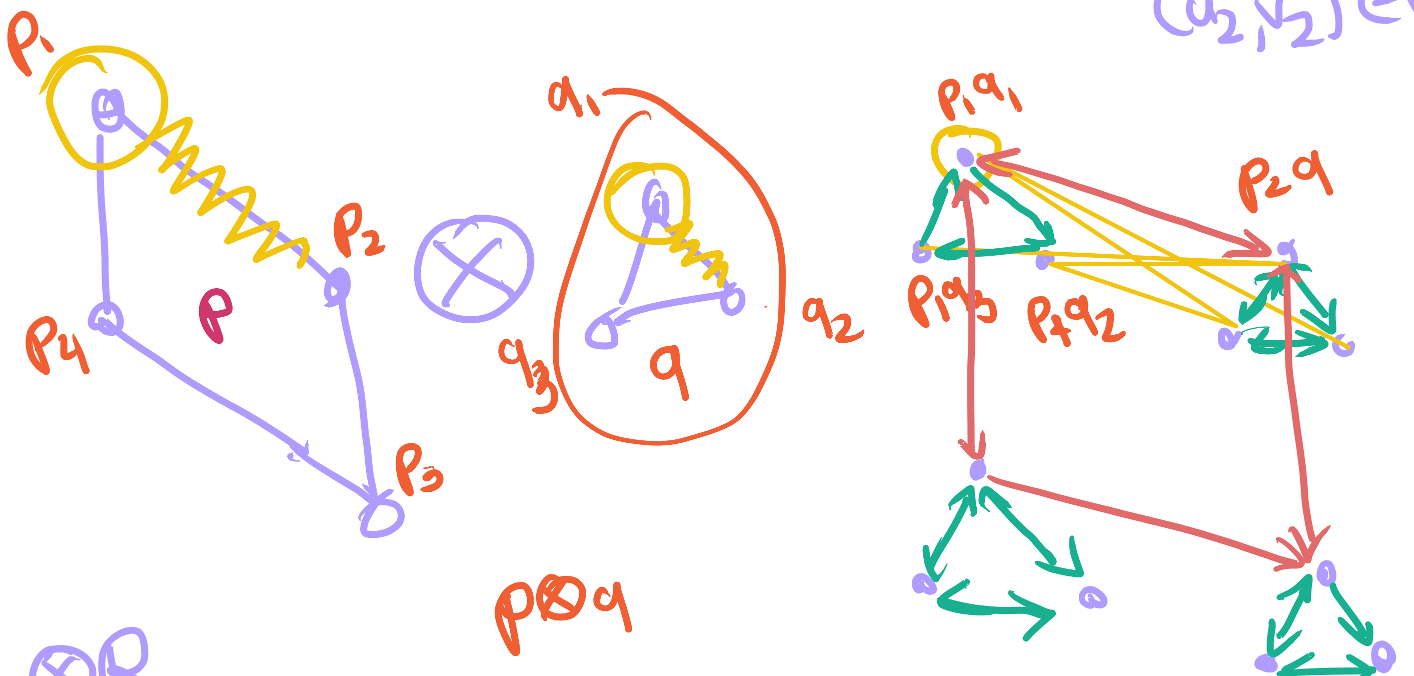
$R_1 \leftarrow$ random w $\quad$ R_2 random W

$G_1 \otimes G_2$

vertices: $\{(v_1, v_2) : v_1 \in V_1 , v_2 \in V_2\}$

edges: "$E_1 \times E_2$"

$= \{(u_1,u_2) \to (v_1,v_2)$ if $(u_1,v_1) \in V_1$, $(u_2,v_2) \in V_2\}$



$p \otimes q$

$R_1 \otimes R_2$

$(x_1 \otimes x_2)_{ij} = x_1(i)\, x_2(j)$

$p \in \Delta^{V_1}$, $q \in \Delta^{V_2}$

$(p \otimes q) \in \Delta^{V_1 \times V_2}$

$(p \otimes q)_{ij} = p_i q_j \geq 0$

$\sum_{ij} p_i q_j =$

$(\sum_i p_i)(\sum_j q_j) = 1$

$p =$ station. dist. on G_1

$q =$ station. dist on G_2

$(p \otimes q)$ station. for $G_1 \otimes G_2$

Proof

$$(R_1 \otimes R_2) = (R_1 \otimes I)(I \otimes R_2)$$

$$(I \otimes R_2)(p \otimes q) = (p \otimes R_2 q) = (p \otimes q)$$

$$(R_1 \otimes I)(p \otimes q) = (R_1 p \otimes q) = (p \otimes q)$$

$$(R_1 \otimes R_2)(x \otimes y) = (\underline{R_1 x \otimes R_2 y})$$

Theorem

① $G_1 \otimes G_2$ is $(d_1 d_2)$-regular, undirected

② eigenvectors/eigenvalues

$(x_1, \lambda_1) \to R_1$

$(x_2, \lambda_2) \to R_2$

$\Rightarrow (x_1 \otimes x_2, \lambda_1 \lambda_2) \to R_1 \otimes R_2$

$$(R_1 \otimes R_2)(x_1 \otimes x_2) = R_1 x_1 \otimes R_2 x_2 = \lambda_1 x_1 \otimes \lambda_2 x_2$$

$$= \lambda_1 \lambda_2 (x_1 \otimes x_2)$$

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let $u_1, \dots, u_{n_1}$ be orthon. eigenv. of R_1

$v_1, \dots, v_{n_2}$ be orth. eigenv. of R_2

then: $\{u_i \otimes v_j\}$. orth. eigenv. of $R_1 \otimes R_2$

$$\langle (u_1 \otimes v_2), (u_4 \otimes v_5) \rangle = \langle u_1, u_4 \rangle \cdot \langle v_2, v_5 \rangle$$

$$\underline{(I \otimes R_H) Z (I \otimes R_H)} \qquad u_1 = \frac{1}{\sqrt{d}} \in \mathbb{R}^d$$

$$R_H = \underbrace{\frac{1}{d} \mathbb{1} \otimes \mathbb{1}}_{S} + \lambda_2 u_2 \otimes u_2 + \sim + \lambda_n u_n \otimes u_n$$

$$R_H = S + \lambda_2 u_2 \otimes u_2 + \cdots + \lambda_n u_n \otimes u_n$$

$R'_H = (1-\gamma_H) S + \lambda_2 + \cdots$ $\quad \lambda_2, \sim, \lambda_n \in [1-\gamma_H, \gamma_H - 1]$

$$R_H' = R - \gamma_H S$$

all eigenvalues of $R_H' \in [1-\gamma_H, \gamma_H - 1]$

$$(I \otimes R_H) Z (I \otimes R_H) \qquad R_H = \gamma_H S + R_H'$$

$$(I \otimes (\gamma_H S + R_H')) Z (I \otimes (\gamma_H S + R_H'))$$

$$((I \otimes \gamma_H S) + (I \otimes R_H')) Z ((I \otimes \gamma_H S) + (I \otimes R_H'))$$

$$\boxed{(I \otimes (A+B)) = (I \otimes A) + (I \otimes B)}$$

$$= \gamma_H^2 \boxed{(I \otimes S) Z (I \otimes S)} = R_G \otimes S$$

$$+ \gamma_H \left((I \otimes S) Z (I \otimes R_H') + (I \otimes R_H') Z (I \otimes S) \right)$$

$$+ (I \otimes R_H') Z (I \otimes R_H')$$

$x \in \mathbb{R}^{V \times d}$

$\|x\| = 1$

$\langle x, \mathbb{1} \rangle = 0$

$$|\langle x, (I \otimes S) Z (I \otimes R_H') x \rangle|$$

$$\leq \underbrace{\|x\|}_{1} \left[\underbrace{\|(I \otimes S) Z (I \otimes R_H') x\|} \right]$$

$$\|(I \otimes S) y\|^2 = \langle (I \otimes S) y, (I \otimes S) y \rangle$$

$$= \langle y, (I \otimes S)^2 y \rangle$$

$\le \|y\|^2 \cdot \lambda_{max}(I \otimes S)^2$

$$\left[\underbrace{\|(I\otimes S)Z(I\otimes R_H')x\|}\right]$$

$$\le \underbrace{|\lambda|_{max}(I\otimes S)}_{\le 1} \; \underbrace{|\lambda|_{max}(Z)}_{\le 1} \; \underbrace{|\lambda|_{max}(I\otimes R_H')}_{=1-\gamma_H}$$

$1-\gamma_G$

$1-\gamma_H$

$$= \gamma_H^2 (I\otimes S)Z(I\otimes S)$$

$$+ \gamma_H \left((I\otimes S)Z(I\otimes R_H') + (I\otimes R_H')Z(I\otimes S) \right)$$

$$+ \underbrace{(I\otimes R_H')Z(I\otimes R_H')}_{(1-\gamma_H)^2}$$

$[1-\gamma_G, \gamma_G - 1]$

$(1, \lambda_2, \ldots, \lambda_n)$ $(1, 0, 0, 0 \ldots)$

$$(I\otimes S)Z(I\otimes S) = \underbrace{R_G \otimes S}_{(1, \lambda_2, \ldots, \lambda_n, 0, \ldots, 0)}$$

$$\underbrace{\lambda_2, \ldots, \lambda_n, 0, \ldots, 0}_{[1-\gamma_G, \gamma_G - 1]}$$